# Quadratic forms and eigenvalues 1

Given the following quadratic forms, we request:

1. Write each quadratic form in matrix form.
2. Compute the eigenvalues of the associated matrix.

## a) In $\mathbb{R}^2$

$$\phi(x_1, x_2) = 4\,x_1^2 + 4\,x_1\,x_2 + 7\,x_2^2.$$

## b) In $\mathbb{R}^2$

$$\phi(x_1, x_2) = x_1^2 + 2\,x_1\,x_2 + x_2^2.$$

## c) In $\mathbb{R}^3$

$$\phi(x_1, x_2, x_3) = 2\,x_1^2 + 4\,x_1\,x_2 + 2\,x_2^2 - 3\,x_3^2.$$

# Solution

## a)

Let the quadratic form be

$$\phi : \mathbb{R}^2 \to \mathbb{R}, \quad \phi(x_1, x_2) \;=\; 4\,x_1^2 \;+\; 4\,x_1\,x_2 \;+\; 7\,x_2^2.$$

### 1) Matrix form

To express $\phi$ in matrix form, we seek a symmetric matrix $Q$ such that

$$\phi(x_1, x_2) \;=\; \begin{pmatrix} x_1 & x_2 \end{pmatrix} \, Q \, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Notice that the term $4\,x_1\,x_2$ corresponds to $2\,q_{12}\,x_1\,x_2$. Therefore, if $q_{11} = 4$ and $q_{22} = 7$, then $2\,q_{12} = 4$ implies $q_{12} = 2$. The associated matrix is thus

$$Q = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}.$$

Hence,

$$\phi(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

### 2) Eigenvalues of the associated matrix

To find the eigenvalues of $Q$, we solve

$$\det(Q - \lambda I) = 0.$$

This is equivalent to

$$\det \begin{pmatrix} 4-\lambda & 2 \\ 2 & 7-\lambda \end{pmatrix} = (4-\lambda)(7-\lambda) - (2)(2) = 0.$$

Expanding,

$$(4-\lambda)(7-\lambda) - 4 = (28 - 4\lambda - 7\lambda + \lambda^2) - 4 = \lambda^2 - 11\lambda + 24.$$

We seek the roots of the polynomial $\lambda^2 - 11\lambda + 24$. Observing that

$$\lambda^2 - 11\lambda + 24 = (\lambda - 8)(\lambda - 3),$$

we obtain

$$\lambda_1 = 8 \quad \text{and} \quad \lambda_2 = 3.$$

## b)

Let the quadratic form be

$$\phi : \mathbb{R}^2 \to \mathbb{R}, \quad \phi(x_1, x_2) \;=\; x_1^2 \;+\; 2\,x_1\,x_2 \;+\; x_2^2.$$

## 1) Matrix form

To express $\phi$ in matrix form, we seek a symmetric matrix $Q$ such that

$$\phi(x_1, x_2) \;=\; \begin{pmatrix} x_1 & x_2 \end{pmatrix} Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Notice that the term $2\,x_1\,x_2$ corresponds to $2\,q_{12}\,x_1\,x_2$. Therefore, if the coefficient of $x_1^2$ is 1 (i.e., $q_{11} = 1$) and the coefficient of $x_2^2$ is 1 (i.e., $q_{22} = 1$), we must have

$$2\,q_{12} = 2 \quad \Longrightarrow \quad q_{12} = 1.$$

Thus, the associated matrix is

$$Q \;=\; \begin{pmatrix} 1 & 1 \\ & \\ 1 & 1 \end{pmatrix}.$$

Hence,

$$\phi(x_1, x_2) \;=\; \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

## 2) Eigenvalues of the associated matrix

To find the eigenvalues of $Q$, we solve

$$\det(Q - \lambda I) = 0.$$

This is equivalent to

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \;=\; (1-\lambda)^2 - 1 \;=\; 0.$$

Expanding the equation:

$$(1-\lambda)^2 - 1 \;=\; 1 - 2\lambda + \lambda^2 - 1 \;=\; \lambda^2 - 2\lambda,$$

which simplifies to

$$\lambda(\lambda - 2) = 0.$$

Thus, the eigenvalues are:

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 2.$$

# c)

Let the quadratic form be

$$\phi : \mathbb{R}^3 \to \mathbb{R}, \quad \phi(x_1, x_2, x_3) \;=\; 2\,x_1^2 \;+\; 4\,x_1\,x_2 \;+\; 2\,x_2^2 \;-\; 3\,x_3^2.$$

## 1) Matrix form

To express $\phi$ in matrix form, we seek a symmetric matrix $Q$ such that

$$\phi(x_1, x_2, x_3) \;=\; \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} Q \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We observe that:

- The coefficient of $x_1^2$ is 2, so $q_{11} = 2$.
- The coefficient of $x_2^2$ is 2, so $q_{22} = 2$.

- The coefficient of $x_3^2$ is $-3$, so $q_{33} = -3$.
- The term $4\,x_1\,x_2$ corresponds to $2\,q_{12}\,x_1\,x_2$, hence $q_{12} = 2$.
- No mixed terms involving $x_3$ appear, so $q_{13} = q_{23} = 0$.

Thus, the associated matrix is

$$Q = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Hence,

$$\phi(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

## 2) Eigenvalues of the associated matrix

To find the eigenvalues of $Q$, we solve

$$\det(Q - \lambda I) = 0.$$

Notice that $Q$ has a block-diagonal form:

$$Q = \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \\ \hline 0 & 0 & -3 \end{array}\right).$$

The $2 \times 2$ block is

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

and the $1 \times 1$ block is $-3$.

The eigenvalues of the $2 \times 2$ block are found from

$$\det \begin{pmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 4 = \lambda^2 - 4\lambda.$$

Thus,

$$\lambda(\lambda - 4) = 0,$$

which gives the eigenvalues:

$$\lambda = 0 \quad \text{and} \quad \lambda = 4.$$

The third eigenvalue, corresponding to the $1 \times 1$ block, is

$$\lambda = -3.$$

Therefore, the eigenvalues of $Q$ are:

$$\lambda_1 = 4, \quad \lambda_2 = 0, \quad \lambda_3 = -3.$$